HYPER VERSION

Received: *August 9, 2001*, Accepted: *October 16, 2001*

String field theory and brane superpotentials

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Abstract: I discuss tree-level amplitudes in cubic topological string field theory, showing that a certain family of gauge conditions leads to an A_∞ algebra of tree-level string products which define a potential describing the dynamics of physical states. Upon using results of modern deformation theory, I show that the string moduli space admits two equivalent descriptions, one given in standard Maurer-Cartan fashion and another given in terms of a 'homotopy Maurer-Cartan problem', which describes the critical set of the potential. By applying this construction to the topological A and B models, I obtain an intrinsic formulation of 'D-brane superpotentials' in terms of string field theory data. This gives a prescription for computing such quantities to all orders, and proves the equivalence of this formulation with the fundamental description in terms of string field moduli. In particular, it clarifies the relation between the Chern-Simons/holomorphic Chern-Simons actions and the superpotential for A/B-type branes.

Keywords: Topological Field Theories, Chern-Simons Theories, D-branes, Superstring Vacua.

JHEP10(2001)018

Contents

1. Introduction

An important subject in D-brane geometry concerns the computation of brane superpotentials, as discussed for the first time in [1]. In typical examples, one is interested in a D-brane wrapping a supersymmetric cycle of a Calabi-Yau threefold, and attempts to describe its moduli space through the critical set of such a quantity.

It is fair to say that results in this direction have remained somewhat imprecise. Part of this lack of precision is due to our incomplete understanding of the mirror map for open string backgrounds. Another reason can be found in the absence of a rigorous formulation of the problem. There are at least two issues to be addressed before one can gain a better understanding:

1. The current definition of 'D-brane superpotentials' is based on an indirect construction involving partially wrapped branes which fill the four uncompactified dimensions. Due to standard difficulties with flux conservation, this is in fact physically inconsistent unless one restricts to non-compact situations (which in themselves have limited physical relevance) or posits some unspecified orientifold constructions which would solve the difficulty.

2. A perhaps more serious problem is the lack of a precise formulation of the relation between flat directions for the superpotential and the string theory moduli space – this is currently resolved by *assuming* that the two spaces coincide, since moduli problems only involve low energy dynamics. However, the fundamental description of D-brane moduli is through the associated string field theory (see section 3.4 of this paper), which in our case is the string field theory of topological A or B type models (the fact that topological string theory suffices follows from the results of [2]). It is not immediately clear how the string field point of view relates to the approach advocated in [1].

The purpose of the present note is to initiate a more thorough analysis of these issues by addressing the two problems above. Our approach is based on a string field theoretic point of view, which was advocated in a wider context in [3, 4] (see also [5] and [6, 7]). This has the advantage that it provides an *intrinsic* description of D-brane moduli spaces. As we shall show below, string field theory allows for a precise formulation of brane 'superpotentials' in a manner which does not require the introduction of partially wrapped branes. Instead, we shall identify the superpotentials of [1] with a generating function for a collection of tree-level string amplitudes computed in a certain gauge. The construction allows us to *prove* that the string field moduli space can be described as the critical set of this function, divided by an appropriate group action. This serves to clarify the relation between the two descriptions, and sheds some light on the connection of D-brane superpotentials to certain mathematical constructions involved in the homological mirror symmetry conjecture [8].

The note is organized as follows. In section 2, we explain the construction of our potential W as a 'generating function' of tree level string amplitudes, and show that the coefficients of its expansion can be expressed in terms of a collection of tree level products which satisfy the constraints of an A_∞ algebra. While we shall apply our constructions to the ungraded A/B models only, the discussion of this (and the next) section is given in slightly more abstract terms, and can be applied to more general situations, such as the graded string field theories of [4]–[7]. In section 3, we discuss two formulations of the brane moduli space, which result by considering the string field equations of motion or the critical manifold of the potential, and dividing by appropriate symmetries. While the first description involves the well-known Maurer-Cartan equation (and the string field theory gauge group $\mathcal{G}$), the second leads to more complicated data, due to the presence of higher order terms. We use the algebraic structure of tree level products to give a mathematically precise formulation of the second description in terms of a homotopy version of the Maurer-Cartan equation and a certain effective symmetry algebra. More precisely, we show that the critical point condition for W can be expressed in terms of a strong homotopy Lie, or L_∞ algebra, the so-called *commutator algebra* of the A_∞ algebra of tree level products. The resulting *homotopy Maurer-Cartan equation* has a symmetry algebra g_W which plays the role of an effective, or 'low energy' remnant of the string field theory gauge algebra. Two solutions of the homotopy Maurer-Cartan equation are identified if they are related by the action of g_W.

The 'homotopy Maurer-Cartan problem' was studied in recent mathematical work of M. Kontsevich [9, 10] and S. Merkulov [11, 12]. Upon combining their results with a simple property of taking commutators (proved in appendix A), we show that the two deformation problems (Maurer-Cartan and homotopy-Maurer-Cartan) give equivalent descriptions of the same moduli space. This explains the relation between the string field theory approach and the low energy point of view advocated in [1]. The fact that a homotopy version of the Maurer-Cartan equation arises naturally in our context suggests a deeper relation between the 'derived deformation program' of [13] and string field theory. It also sheds light on the relation between D-brane superpotentials and the abstract methods currently used in the homological mirror symmetry literature. In section 4, we apply our construction to the topological B/A models which describe D-branes wrapping a Calabi-Yau manifold, respectively one of its three-cycles, thereby obtaining an explicit all order construction of the associated D-brane superpotentials (for the A model we only consider the large radius limit, since the string field theory at finite radius requires a more sophisticated analysis [14, 15, 16]). Section 5 presents our conclusions. Appendix A collects some facts about homotopy associative and homotopy Lie algebras and their deformation theory and proves a result needed in the body of the paper. Appendix B contains the details of a calculation relevant for understanding the effective symmetry algebra.

For the mathematically-oriented reader, I mention that many of the arguments used in this paper are adaptations of results known in the homological mirror symmetry literature. Unfortunately, they do not seem to have been integrated with each other and with the string field theory perspective, which is why their physical significance has remained somewhat obscure.

2. Tree level potentials in open string field theory

We start by presenting a method for computing tree level potentials in (cubic) open string field theory, and analyze the result in terms of A_∞ algebras. More precisely, we show that a certain gauge-fixing procedure leads to a collection of tree level string products which satisfy the constraints of an A_∞ algebra as well as certain cyclicity properties. The construction we use is intimately related to the work of [17] and [10], and we shall borrow some of their results, with certain modifications. While we shall later apply this construction to topological A/B strings (see section 4), we chose to present it in an abstract form in order to display the complete similarity between the two theories. The procedure discussed below can also be applied to more general models, such as the graded string field theories of [4]–[7].

2.1 The abstract model

Let us start with a cubic (topological) string field theory based on the action:

$$S[\phi] = \frac{1}{2}\langle\phi, Q\phi\rangle + \frac{1}{3}\langle\phi, \phi\bullet\phi\rangle\,, \tag{2.1}$$

where the string field ϕ is a degree one element of the boundary space $\mathcal{H}$, a $\mathbf{Z}$-graded differential associative algebra with respect to the (degree one) BRST operator Q, the boundary string product $\bullet$ and the worldsheet degree[1] $|\cdot|$. Since we shall deal with a single boundary sector (a single D-brane), we do not need to consider a category structure on $\mathcal{H}$ as in [3, 4]. Our arguments can be generalized to that case, but in this note I wish to keep things simple. Remember from [18, 19, 3] that the non-degenerate bilinear form $\langle .,.\rangle$ is invariant with respect to Q and the boundary product:

$$\langle Qu, v\rangle = -(-1)^{|u|}\langle u, Qv\rangle\,, \qquad \langle u\bullet v, w\rangle = \langle u, v\bullet w\rangle\,. \tag{2.2}$$

[1] For the topological A/B models, $|\cdot|$ is the charge with respect to the anomalous U(1) current on the worldsheet.

It also has the graded symmetry property:

$$\langle v, u\rangle = (-1)^{|u||v|}\langle v, u\rangle\,, \tag{2.3}$$

and obeys the selection rule:

$$\langle u, v\rangle = 0 \qquad \text{unless } |u| + |v| = 3\,. \tag{2.4}$$

Due to this selection rule, the sign factor $(-1)^{|u||v|}$ in equation (2.3) can always be taken to be $+1$.

2.2 Gauge-fixing data

We further assume that we are given a positive-definite hermitean product h on $\mathcal{H}$, which is antilinear with respect to its first variable and couples only states of equal worldsheet degree:

$$h(u, v) = 0 \qquad \text{unless } |u| = |v|\,. \tag{2.5}$$

We let Q^+ be the hermitean conjugate of Q with respect to h:

$$h(Qu, v) = h(u, Q^+v)\,. \tag{2.6}$$

Note that Q^+ is nilpotent and homogeneous of degree -1.

Let us define an antilinear operator c on $\mathcal{H}$ through the relation:

$$h(u, v) = \langle cu, v\rangle\,. \tag{2.7}$$

Since $\langle cu, v\rangle$ vanishes unless $|cu| + |v| = 3$, while $h(u, v)$ vanishes unless $|v| - |u| = 0$, we must have:

$$|cu| = 3 - |u| \tag{2.8}$$

on homogeneous elements u.

Hermicity of h is then equivalent with the property:

$$\overline{\langle cu, v\rangle} = \langle cv, u\rangle\,, \tag{2.9}$$

which is easily seen to imply:

$$\langle c^2u, v\rangle = \langle u, c^2v\rangle\,. \tag{2.10}$$

We shall *assume* that the metric h is chosen such that $c^2 = Id$ (this is possible in the topological A/B models, as we shall see in section 4). With this hypothesis, it is easy to see that c is an antilinear isometry with respect to h:

$$h(cu, cv) = h(v, u)\,, \tag{2.11}$$

and that the operator Q^+ can be expressed as:

$$Q^+u = (-1)^{|u|}cQcu\,. \tag{2.12}$$

Indeed, one has:

$$(-1)^{|u|}h(cQcu,v) = (-1)^{|u|}\langle Qcu, v\rangle = (-1)^{1+|u|+|cu|}\langle cu, Qv\rangle = \langle cu, Qv\rangle = h(u, Qv)\,. \tag{2.13}$$

Using (2.12), one can check that the defining relation for Q^+ (namely $h(Qu,v) = h(u, Q^+v)$) implies:

$$\langle Q^+u, v\rangle = (-1)^{|u|}\langle u, Q^+v\rangle\,. \tag{2.14}$$

This property will be essential in subsection 2.5.

2.3 The propagator

The string field action (2.1) has the gauge symmetry:

$$\phi \to \phi - Q\alpha - [\phi, \alpha]\,, \tag{2.15}$$

with α a degree zero element of $\mathcal{H}$. We are interested in partially fixing this symmetry through the gauge condition:[2]

$$Q^+\phi = 0\,. \tag{2.16}$$

A thorough analysis of gauge fixing would generally require the full machinery of the BV formalism, but luckily we will not need this here. In fact, we shall only be interested in tree level scattering amplitudes for the topological A/B models, for which it suffices to understand the relevant propagator. That the BV analysis does not modify the discussion in this case follows, for example, from the work of [20].

For this purpose it is convenient to consider the 'Hodge theory' of Q. Let us define:[3]

$$H = [Q, Q^+] = QQ^+ + Q^+Q\,, \tag{2.17}$$

and let K denote the kernel of H. As usual in Hodge theory, one has:

$$\begin{aligned} K &= \operatorname{Ker} Q \cap \operatorname{Ker} Q^+\,, & \mathcal{H} &= K \oplus \operatorname{Im} Q \oplus \operatorname{Im} Q^+\,, \\ \operatorname{Ker} Q &= K \oplus \operatorname{Im} Q\,, & \operatorname{Ker} Q^+ &= K \oplus \operatorname{Im} Q^+\,, \end{aligned} \tag{2.18}$$

where the direct sums involved are orthogonal with respect to h. The operator H is hermitean and commutes with Q and Q^+, and thus its restriction to $K^\perp = \operatorname{Im} Q \oplus \operatorname{Im} Q^+$ gives an automorphism of this space. We shall denote the inverse of $H|_{K^\perp}$ by $\frac{1}{H}$. It is easy to check that the operators $\pi_Q = Q\frac{1}{H}Q^+$ and $\pi_{Q^+} = Q^+\frac{1}{H}Q$ are orthogonal projectors on $\operatorname{Im} Q$ and $\operatorname{Im} Q^+$. It follows that the operator:

$$P = 1 - \left(Q^+\frac{1}{H}Q + Q\frac{1}{H}Q^+\right) \tag{2.19}$$

is the orthogonal projector on K.

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[2] For the topological A/B models discussed in section 4, this is essentially the Siegel gauge (as follows from the fact that in such models one can identify b_0 with Q^+ for an appropriate choice of the metric h), hence the associated string field correlators admit a direct interpretation as string scattering amplitudes.

[3] In this note, $[.,.]$ always stands for the graded commutator.

To find the relevant propagator, one must identify a maximal subspace of $\mathcal{H}^1$ on which the quadratic form $S_{\rm kin}(\phi) = \langle \phi, Q\phi \rangle$ is non-degenerate. Since the BPZ form $\langle .,. \rangle$ is non-degenerate, the kernel of the bilinear symmetric form $\langle \phi, Q\psi \rangle = \langle \psi, Q\phi \rangle = \langle Q\phi, \psi \rangle = h(cQ\phi, \psi) = -h(Q^+(c\phi), \psi)$ on $\mathcal{H}^1$ (the polar form of $S_{\rm kin}$) is $({\rm Im}\, Q^+)^\perp \cap \mathcal{H}^1 = {\rm Ker}\, Q \cap \mathcal{H}^1$. Hence a maximal subspace with the desired property is $({\rm Ker}\, Q)^\perp \cap \mathcal{H}^1 = {\rm Im}\, Q^+ \cap \mathcal{H}^1$. The restriction of Q to ${\rm Im}\, Q^+$ gives an isomorphism from ${\rm Im}\, Q^+$ to ${\rm Im}\, Q$, whose inverse we denote by $1/Q$. It follows that for $\phi \in {\rm Im}\, Q^+ \cap \mathcal{H}^1$ we can write:

$$\langle \phi, Q\phi \rangle = \left\langle \psi, \frac{1}{Q}\psi \right\rangle , \tag{2.20}$$

where $\psi = Q\phi$. Hence the pull-back of $S_{\rm kin}$ through the map $1/Q : {\rm Im}\, Q \cap \mathcal{H}^2 \stackrel{\approx}{\rightarrow} {\rm Im}\, Q^+ \cap \mathcal{H}^1$ can be identified with its 'inverse'.

Let us consider a 'Green operator' G for H, which satisfies:

$$HG = 1 - P = \pi_Q + \pi_{Q^+} . \tag{2.21}$$

We shall choose the solution $G = \frac{1}{H}(1 - P) = \frac{1}{H}(\pi_Q + \pi_{Q^+})$. We next define $U = Q^+G = Q^+\frac{1}{H}(\pi_Q + \pi_{Q^+}) = Q^+\frac{1}{H}\pi_Q = \frac{1}{H}Q^+$. Then the projectors π_Q, π_{Q^+} can be written as:

$$\pi_Q = QU , \qquad \pi_{Q^+} = UQ . \tag{2.22}$$

Hence we can write:

$$U = \frac{1}{Q}\pi_Q . \tag{2.23}$$

It is clear that U plays the role of propagator for the Q-exact modes. Following the terminology of [21], states belonging to the subspace ${\rm Im}\, Q$ will be called *spurious*, while states belonging to $K = {\rm Ker}\, Q \cap {\rm Ker}\, Q^+$ will be called *physical*.[4] The elements of ${\rm Ker}\, Q^\perp = {\rm Im}\, Q^+$ are the *unphysical* states. Relations (2.18) show that the off-shell state space $\mathcal{H}$ decomposes into physical, spurious and unphysical components. It is clear that U propagates spurious states into unphysical states and projects out everything else.

2.4 Tree level amplitudes and the potential

It may seem strange that we are interested in a propagator which describes the dynamics of non-physical states. The reason why such an object is relevant is that *the basic string product $\bullet$ does not map physical states into physical states.* Indeed, the string field theory axioms assure that $\bullet$ maps ${\rm Ker}\, Q \times {\rm Ker}\, Q$ into ${\rm Ker}\, Q$ (since the BRST operator acts as a derivation of the product), but it is *not* true, in general, that $\bullet$ maps ${\rm Ker}\, Q^+ \times {\rm Ker}\, Q^+$ into ${\rm Ker}\, Q^+$ (in particular, Q^+ does not act as a derivation of the string product, even though it has property (2.14) with respect to

[4] Strictly speaking, physical states are *degree one* elements of K, but we shall sometimes use the words 'physical states' to mean states belonging to K. We hope this does not lead to confusion.

the bilinear form). If one considers a two-string joining process $(u_1, u_2) \rightarrow v = u_1 \bullet u_2$, then the state v will generally not satisfy the gauge-fixing condition $Q^+ v = 0$, even if both u_1 and u_2 belong to the space of physical states K^1. Since $Qv = 0$, we have $v \in \operatorname{Ker} Q = K \oplus \operatorname{Im} Q$, so the precise way in which the gauge condition is violated is that v may acquire a component $v_Q \in \operatorname{Im} Q$ along the subspace of spurious states. This component then propagates into the unphysical state $Uv_Q = Uv \in \operatorname{Im} Q^+$. If the composite string now interacts with an open string in the state $u_3 \in K^1$, the result is $(Uv) \bullet u_3 = (U(u_1 \bullet u_2)) \bullet u_3$, which can be measured by projecting onto K etc. It follows that string amplitudes $\langle\langle u_1 \dots u_n \rangle\rangle$, where $u_1 \dots u_n$ are (degree one) physical states, are built according to the Feynman rules of the cubic theory (2.1) upon using the propagator U. To be precise, we define $\langle\langle u_1 \dots u_n \rangle\rangle^{(n)}$ to be *amputated* amplitudes, so there are no insertions of propagators on the external legs. Moreover, we shall only be interested in tree level correlators, which we denote by $\langle\langle u_1 \dots u_n \rangle\rangle^{(n)}_{\rm tree}$.

We next define a tree-level potential by summing over all (signed) amputated (not necessarily connected) tree-level scattering amplitudes with at least three legs:

$$W[\phi] = \sum_{n \geq 3} \frac{1}{n} (-1)^{n(n-1)/2} \langle\langle \phi, \dots, \phi \rangle\rangle^{(n)}_{\rm tree} \,. \tag{2.24}$$

The string field ϕ in this expression belongs to the space $K^1 = K \cap \mathcal{H}^1$.

It is convenient formulate this in an algebraic manner. Let us define string products $r_n : K^{\otimes n} \rightarrow K$ $(n \geq 2)$ by following the tree level Feynman diagrams of our theory, but applied to propagation of arbitrary states $u \in K$ (i.e. we formally allow u to have degree different from one). Upon following the combinatorics of tree level diagrams, one can easily check that r_n can be described as follows (figure 1):

1. We first define products $\lambda_n : \mathcal{H}^n \rightarrow \mathcal{H}$ through $\lambda_2 = \bullet$ and the recursion relation:

$$\begin{aligned} \lambda_n(u_1, \dots, u_n) = {} & (-1)^{n-1} (U\lambda_{n-1}(u_1, \dots, u_{n-1})) u_n - \\ & - (-1)^{n|u_1|} u_1 (U\lambda_{n-1}(u_2, \dots, u_n)) - \\ & - \sum_{\substack{k+l=n \\ k,l \geq 2}} (-1)^{k+(l-1)(|u_1|+\dots+|u_k|)} (U\lambda_k(u_1, \dots, u_k)) \times \\ & \qquad \times (U\lambda_l(u_{k+1}, \dots, u_n)) \,, \end{aligned} \tag{2.25}$$

   for $u_1 \dots u_n$ in $\mathcal{H}$.

2. The products r_n are then given by:

$$r_n(u_1, \dots, u_n) = P\lambda_n(u_1, \dots, u_n) \,, \tag{2.26}$$

   for $u_1, \dots, u_n \in K$.

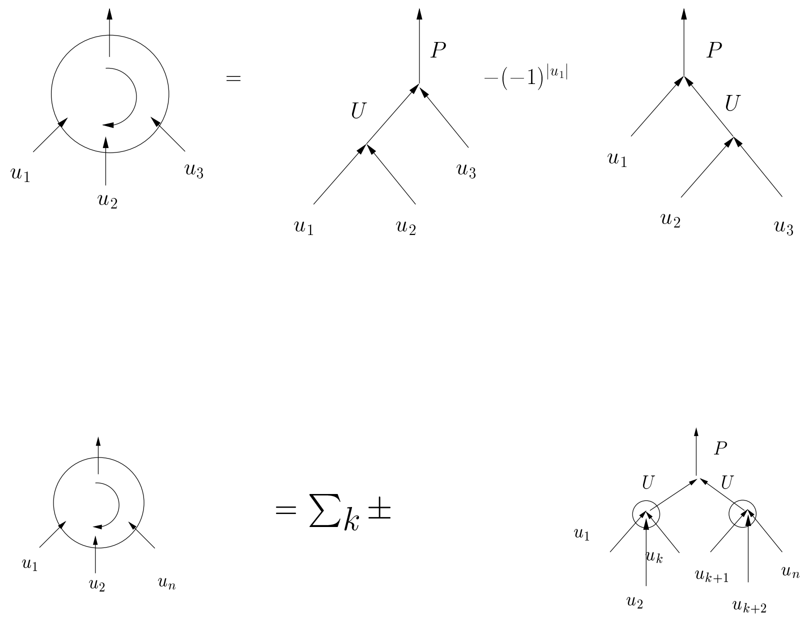

Figure 1: Expressing disk string correlators of physical states in terms of Feynman rules. The upper figure shows the case of the product $r_3 = P\left[U(u_1 \bullet u_2) \bullet u_3 - (-1)^{|u_1|} u_1 \bullet U(u_2 \bullet u_3)\right]$. The lower figure shows the general decomposition of λ_n with respect to the products λ_k $(k < n)$.

The recursion relation (2.25) describes the decomposition of an order n tree level product into lower order products, as explained in figure 1. This encodes the combinatorics of tree level Feynman diagrams. With our conventions for the grading, the product r_n has degree $2 - n$:

$$|r_n(u_1 \dots u_n)| = |u_1| + \dots |u_n| + 2 - n\,. \tag{2.27}$$

Products of the type (2.26) were considered in [17], [22] and [10]. In those papers, it is shown that they define an A_∞ algebra structure on K, i.e. they satisfy the following constraints:

$$\sum_{\substack{k+l=n+1 \\ j=0\dots k-1}} (-1)^s r_k(u_1 \dots u_j, r_l(u_{j+1} \dots u_{j+l}), u_{j+l+1} \dots u_n) = 0\,, \tag{2.28}$$

for all $n \geq 3$, where $s = l(|u_1| + \dots |u_j|) + j(l-1) + (k-1)l$. Note that our algebra has $r_1 = 0$. Some basic facts about A_∞ algebras are collected in appendix A.

With these preparations, we define (extended) tree level amplitudes by:

$$\langle\langle u_1 \dots u_n \rangle\rangle^{(n)}_{\text{tree}} = \langle u_1, r_{n-1}(u_2 \dots u_n)\rangle\,, \tag{2.29}$$

where $u_1 \dots u_n$ belong to K. Expression (2.29) makes mathematical sense for elements of K of arbitrary degree. With our definition of r_n, the quantities (2.29) coincide with the amputated tree level amplitudes when $u_1 \dots u_n$ are degree one elements of K. Hence we can write our potential as follows:

$$W(\phi) = \sum_{n\geq 2} \frac{1}{n+1}(-1)^{n(n+1)/2}\langle \phi, r_n(\phi^{\otimes n})\rangle \, . \tag{2.30}$$

Observation. In a topological string theory, the potential W can be identified with a certain representation of the open string analogue of the 'free energy' of [23], which was studied from a deformation-theoretic perspective in [24]. To understand this relation, let us pick a basis $\phi_1 \dots \phi_h$ of K^1 and look for the value of our potential at the point $\phi = \sum_{j=1}^{h} t_j \phi_j$, where $t = (t_1, \dots, t_h)$ is a collection of parameters:

$$W(t) = \sum_{s_1 \dots s_h \geq 0} t_1^{s_1} \dots t_h^{s_h} W_{s_1 \dots s_h}(\phi_1 \dots \phi_h) \, . \tag{2.31}$$

Here $W_{s_1 \dots s_h}(\phi_1 \dots \phi_h)$ involves a sum of string amplitudes with s_j insertions of ϕ_j in all possible orderings on the disk's boundary. Upon fixing the position of three distinct insertion points, those coefficients $W_{s_1 \dots s_h}$ for which $s_1 + \dots + s_h \geq 3$ can be formally written in terms of integrated descendants. It is then easy to see that this corresponds to the boundary potential considered in [24]. Our description differs from that of [24] in that it uses a gauge-fixing prescription in order to give a concrete formula for string amplitudes. As we shall see below, this allows one to give a more detailed analysis of boundary deformations, which goes beyond the infinitesimal (first order) approach. In general, it seems that progress in the study of deformations requires the full force of string field theory.

2.5 Cyclicity

It is possible to show that our tree level correlators satisfy the following cyclicity property:

$$\langle\langle u_1 \dots u_n \rangle\rangle_{\text{tree}}^{(n)} = (-1)^{(n-1)(|u_1|+|u_2|+1)+|u_1|(|u_2|+\dots+|u_n|)} \langle\langle u_2 \dots u_n, u_1 \rangle\rangle_{\text{tree}}^{(n)} \, , \tag{2.32}$$

i.e.:

$$\langle u_1, r_n(u_2 \dots u_{n+1})\rangle = (-1)^{n(|u_1|+|u_2|+1)+|u_1|(|u_2|+\dots+|u_{n+1}|)} \langle u_2, r_n(u_3 \dots u_{n+1}, u_1)\rangle \, . \tag{2.33}$$

For this, note that (2.14) implies that the operator U has a similar property:

$$\langle Uu, v\rangle = (-1)^{|u|}\langle u, Uv\rangle \, . \tag{2.34}$$

The rest of the argument is then formally identical[5] with that given in [17], and will not be repeated here.

[5] The abstract form of the argument of [17] can be most easily recovered upon defining the 'trace' $\mathrm{Tr}(u) := \langle u, 1\rangle = \langle 1, u\rangle$ on $\mathcal{H}$, where 1 is the unit of the boundary algebra $(\mathcal{H}, \bullet)$. Invariance of the bilinear form with respect to the boundary product implies $\langle u, v\rangle = \mathrm{Tr}(u \bullet v)$, which allows one to apply the cyclicity argument of [17] to our more general situation.

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We note that the selection rule for the bilinear form $\langle .,. \rangle$ allows one to simplify (2.33) to:

$$\langle u_1, r_n(u_2 \dots u_{n+1}) \rangle = (-1)^{n(|u_2|+1)} \langle u_2, r_n(u_3 \dots u_{n+1}, u_1) \rangle \,. \tag{2.35}$$

3. Two descriptions of the boundary moduli space

In this section we give two descriptions of the moduli space of vacua. The first is the standard construction in terms of solutions of the string field equations of motion, while the second results by considering extrema of the potential W. We shall show that the two descriptions are locally equivalent by formulating them in terms of Lie/homotopy Lie algebras and using mathematical results of M. Kontsevich and S. A. Merkulov. Some ideas of this section are already implicit in [25].

3.1 The string field theory description

The space $\mathcal{M}$ of vacua of a cubic string field theory can be described as the moduli space of degree one solutions to the Maurer-Cartan equations (= string field equations of motion)

$$Q\phi + \frac{1}{2}[\phi, \phi] = 0 \,, \tag{3.1}$$

taken modulo the action of the gauge group $\mathcal{G}$ generated by transformations of the form:

$$\phi \to \phi - Q\alpha - [\phi, \alpha] \,, \tag{3.2}$$

where the infinitesimal generator α is a degree zero element of $\mathcal{H}$. In these equations, $[.,.]$ stands for the graded commutator in the graded associative algebra $\mathcal{H}$:

$$[u, v] := u \bullet v - (-1)^{|u||v|} v \bullet u \,. \tag{3.3}$$

The gauge group $\mathcal{G}$ can be described globally as follows. When endowed with the commutator (3.3), the space $\mathcal{H}$ becomes a differential graded Lie algebra $\mathbf{g}$; the relation between this and the graded associative algebra $(\mathcal{H}, \bullet)$ is entirely similar to that between a usual (ungraded) associative algebra and the corresponding Lie algebra. It is easy to see that the subspace $\mathcal{H}^0$ of degree zero elements forms an (ungraded) Lie sub-algebra $g := \mathbf{g}^0$ of $\mathbf{g} = (\mathcal{H}, [.,.])$; this coincides with the commutator algebra of the (ungraded) associative subalgebra $(\mathcal{H}^0, \bullet)$. The gauge group $\mathcal{G}$ is formally the Lie group obtained by exponentiating this Lie algebra. It consists of elements $\lambda = \exp(\alpha)$, where exp denotes the exponential map. This description is only formal because, even in the simplest case of topological string theories, the Lie algebra $\mathbf{g}$ is in fact infinite-dimensional, and thus the exponential has to be carefully defined on a case by case basis. The group $\mathcal{G}$ acts on the space $\mathcal{H}$ though the obvious extension of its adjoint representation:

$$e^{\alpha} \bullet u = e^{ad_{\alpha}} u \,, \tag{3.4}$$

where ad is the adjoint action of the Lie algebra g:

$$ad_\alpha(u) = [\alpha, u]\,. \tag{3.5}$$

Under the action of e^α, the string field ϕ is taken to transform as a 'connection':

$$\phi \to \phi^\alpha = e^{ad_\alpha}\phi - \frac{e^{ad_\alpha} - 1}{ad_\alpha} Q\alpha\,, \tag{3.6}$$

where the last term is defined through its series expansion:

$$\frac{e^{ad_\alpha} - 1}{ad_\alpha} = \sum_{n\geq 1} \frac{1}{n!}(ad_\alpha)^{n-1}\,. \tag{3.7}$$

Upon expanding (3.6) to first order in α, one recovers the infinitesimal form (3.2).

3.2 Description through extrema of the potential

The description of the moduli space discussed above displays the complete analogy between cubic string field theory and Chern-Simons field theory. It is possible to give an entirely different construction, which is based on the potential (2.30). Indeed, one can ask for the moduli space $\mathcal{M}_W$ of string field configurations $\phi \in K^1$ which extremize this potential:

$$\frac{\partial W}{\partial \phi}(\phi) = 0 \Leftrightarrow \sum_{n\geq 2} (-1)^{n(n+1)/2} r_n(\phi^{\otimes n}) = 0\,. \tag{3.8}$$

To arrive at this equation, we noticed that the cyclicity property (2.32) implies:

$$\langle\langle u_1 \dots u_n\rangle\rangle^{(n)}_{\text{tree}} = \langle\langle u_2 \dots u_n, u_1\rangle\rangle^{(n)}_{\text{tree}} \qquad \text{for } u_1 \dots u_n \in \mathcal{H}^1\,. \tag{3.9}$$

3.2.1 Formulation of the extremum condition in terms of an L_∞ algebra

In order to understand the relevant algebra of symmetries, it is convenient to rewrite equation (3.8) in terms of the 'commutator algebra' of the A_∞ algebra defined by the products r_n. Just as any (differential) graded associative algebra defines a (differential) graded Lie algebra, any A_∞ algebra has an associated L_∞ (or *strong homotopy Lie*) algebra. To describe this construction, we first recall the definition of these mathematical structures (the reader can find more details in appendix A).

L_∞ algebras. L_∞ algebras $(L, \{m_n\}_{n\geq 1})$ are natural generalizations of Lie algebras, being defined by a countable family of n-tuple products m_n subject to the constraints:

$$\sum_{k+l=n+1} \sum_{\sigma\in Sh(k,n)} (-1)^{k(l-1)}\chi(\sigma; u_1 \dots u_n) m_l(m_k(u_{\sigma(1)} \dots u_{\sigma(k)}), u_{\sigma(k+1)} \dots u_{\sigma(n)}) = 0\,, \tag{3.10}$$

and to the graded antisymmetry condition:

$$m_n(u_{\sigma(1)} \dots u_{\sigma(n)}) = \chi(\sigma; u_1 \dots u_n) m_n(u_1 \dots u_n)\,, \tag{3.11}$$

for any permutation σ of the set $\{1, \dots, n\}$. In these relations, the symbol $\chi(\sigma, u_1 \dots u_n) = \pm 1$ is defined through:

$$\chi(\sigma; u_1 \dots u_n) := \epsilon(\sigma)\epsilon(\sigma; u_1 \dots u_n)\,, \tag{3.12}$$

where $\epsilon(\sigma)$ is the signature of the permutation σ and $\epsilon(\sigma; u_1 \dots u_n)$ is the so-called *Koszul sign*, i.e. the sign obtained when unshuffling the element $u_{\sigma(1)} \odot \dots \odot u_{\sigma(n)}$ to the form $u_1 \odot \dots \odot u_n$ in the free graded commutative algebra $\odot^* L = \oplus_{k \geq 0} \odot^k L$:

$$u_{\sigma(1)} \odot \dots \odot u_{\sigma(n)} = \epsilon(\sigma; u_1 \dots u_n) u_1 \odot \dots \odot u_n\,. \tag{3.13}$$

We remind the reader that $\odot^* L$ is built upon dividing the free associative algebra $\otimes^* L = \oplus_{k \geq 0} \otimes^k L$ through the homogeneous ideal generated by elements of the form $u \otimes v - (-1)^{|u||v|} v \otimes u$.

The sum in (3.10) is over so-called (k, n)-*shuffles*, i.e. permutations σ on n elements which satisfy:

$$\sigma(1) < \sigma(2) < \dots < \sigma(k)\,, \qquad \sigma(k+1) < \sigma(k+2) < \dots < \sigma(n)\,. \tag{3.14}$$

An L_∞ algebra such that $m_n = 0$ for all $n \geq 3$ is simply a differential graded Lie algebra, with the differential $Q = m_1$ and the graded Lie bracket $[.,.] = m_2$.

The commutator algebra of an A_∞ algebra. Given an A_∞ algebra $(A, \{r_n\}_{n \geq 1})$, its *commutator algebra* [26] is the the L_∞ algebra defined on the same underlying space $L = A$ by the products:

$$m_n(u_1 \dots u_n) = \sum_{\sigma \in S_n} \chi(\sigma, u_1 \dots u_n) r_n(u_{\sigma(1)} \dots u_{\sigma(n)})\,, \tag{3.15}$$

where S_n is the permutation group on n elements. It is easy to check by computation that the defining constraints of an L_∞ algebra are satisfied. A more synthetic description of this construction (in terms of so-called bar duals) can be found in [26] and is summarized in appendix A.

The homotopy Maurer-Cartan problem. Let us return to equations (3.8). Performing the commutator construction for our A_∞ algebra (K, r_n), we obtain an L_∞ algebra (K, m_n) whose first product m_1 vanishes. If we apply (3.15) to $u_1 = \dots = u_n = \phi$, we obtain:

$$m_n(\phi \dots \phi) = n! r_n(\phi \dots \phi)\,, \tag{3.16}$$

where we used the fact that $\epsilon(\sigma, \phi \dots \phi) = \epsilon(\sigma)$ (and thus $\chi(\sigma, \phi \dots \phi) = +1$), which

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follows from $|\phi| = 1$. Hence one can rewrite the extremum conditions (3.8) as:

$$\sum_{n\geq 2} \frac{(-1)^{n(n+1)/2}}{n!} m_n(\phi^{\otimes n}) = 0\,. \tag{3.17}$$

This is the standard form of the so-called 'homotopy Maurer-Cartan equation' in an L_∞ algebra [9, 12]. It is possible to check (see appendix B) that equations (3.17) are invariant with respect to infinitesimal transformations of the form:

$$\phi \to \phi' = \phi + \delta_\alpha \phi\,, \qquad \text{with } \delta_\alpha\phi = -\sum_{n\geq 2} \frac{(-1)^{n(n-1)/2}}{(n-1)!} m_n(\alpha \otimes \phi^{\otimes n-1})\,, \tag{3.18}$$

where α is a degree zero element of K.[6] The moduli space $\mathcal{M}_W$ is then defined by modding out the space of solutions to (3.8) or (3.17) through the action of the symmetry algebra g_W generated by (3.18).

Observations. (1) The basic difference between the algebras $g = \mathbf{g}^0$ and g_W is that the action of the former involves the BRST differential (equation (3.2)), while the action of the latter does not. Passage from the string field theory to the tree level effective description 'rigidifies' g to g_W.

(2) The algebra of transformations (3.18) is generally open, i.e. it only closes on the critical set of W (this seems to happen for the graded string field theories of [4]–[7]). For ϕ satisfying the critical point equations (3.8), the commutator $\delta_\alpha\delta_\beta\phi - \delta_\beta\delta_\alpha\phi$ does not generally coincide with $\delta_{[\alpha,\beta]}$, but with an infinitesimal transformation $\delta_\gamma\phi$, where $\gamma = \gamma(\alpha, \beta, \phi)$ is given by a sum over products of the form $m_n(\alpha \otimes \beta \otimes \phi^{n-2})$. This situation is familiar in the context of the BV-BRST formalism. The structure of g_W is much simpler for the ungraded string field theories discussed in section 4. In this case, one can show that the gauge algebra closes away from the critical set of W to a standard Lie algebra. This follows from the argument given below.

3.2.2 A particular case

Let us assume for a moment that:

$$Q^+(\alpha \bullet u) = \alpha \bullet Q^+ u \text{ for } \alpha \in K^0 \text{ and } u \in \mathcal{H} \tag{3.19}$$

(this holds for the topological A/B models, which will be discussed below). With this assumption, one can show that the infinitesimal gauge transformations (3.18) reduce to:

$$\phi \to \phi + [\alpha, \phi]\,. \tag{3.20}$$

To prove this, note that (3.19) implies $U(\alpha \bullet u) = 0$ for any u which belongs to $\mathrm{Ker}\, Q^+$.

[6] These transformations can be formulated abstractly in terms of so-called 'pointed Q-manifolds' [9]. The explicit form given here is justified by the calculations of appendix B. A similar explicit form is written down in [12], though the choice of signs in that paper seems to differ from ours.

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If we consider the diagrammatic expansion of the products r_n, this implies that a connected contribution to $r_n(\alpha, \phi^{\otimes n-1})$ (with $\alpha \in K^0$ and $\phi \in K^1$) vanishes unless both of the following conditions are satisfied:

1. the α-insertion belongs to the highest level of the associated tree (i.e. belongs to the same node as a ϕ-insertion). This follows from the fact that any expression of the form $U(\alpha \bullet Uv)$ vanishes, since $Uv \in \mathrm{Im}\, Q^+ \subset \mathrm{Ker}\, Q^+$. Hence branches of the type displayed in figure 2a are forbidden.

2. its insertion node is the root of the tree, i.e. the node where the projector P is inserted. This follows from the fact that $U(\alpha \bullet \phi) = 0$, since $\phi \in K^1 \subset \mathrm{Ker}\, Q^+$. Hence branches of the type displayed in figure 2b are forbidden.

It is clear that it is impossible to satisfy both conditions (1) and (2) unless $n = 2$, since any tree belonging to the diagrammatic expression of $r_n(\alpha, \phi^{\otimes n-1})$ for $n \geq 3$ must contain at least a branch of the two types depicted in figure 2. It follows that all terms in the sum of (3.18) vanish except for the summand $n = 2$. Since we clearly have $\alpha \bullet \phi \in K^1$ (by virtue of (3.19) and (2.2)), it follows that $r_2(\alpha, \phi) = P(\alpha \bullet \phi) = \alpha \bullet \phi$, and thus $m_2(\alpha, \phi) = [\alpha, \phi]$. This shows that (3.18) reduces to (3.20).

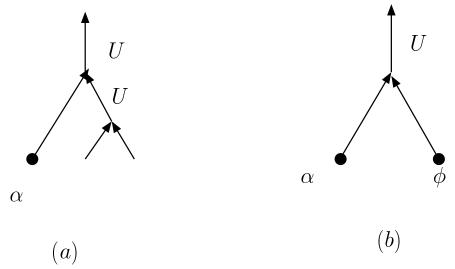

Figure 2: Branches which lead to vanishing of a tree-level contribution to $r_n(\alpha, \phi^{\otimes n-1})$. The two edges on the right of figure (a) may be internal or external.

Relation (3.19) implies that K^0 is an associative subalgebra of $(\mathcal{H}, \bullet)$, and hence $\mathbf{g}^0_W := (K^0, [.,.])$ is a Lie subalgebra of string field gauge algebra $\mathbf{g}^0 = (\mathcal{H}^0, [.,.])$. In particular, the gauge algebra g_W closes away from the critical set of W and can be identified with the Lie algebra $\mathbf{g}^0_W$. Transformation (3.20) integrates to

$$\phi \to \phi^\alpha = e^{ad_\alpha} \phi \,. \tag{3.21}$$

Hence equations (3.8) are invariant with respect to the symmetry group G_W obtained by exponentiating the Lie algebra $g_W \equiv \mathbf{g}^0_W = (K^0, [.,.])$, and $\phi \in K^1$ transforms in its adjoint representation.

3.3 Local equivalence of the two constructions

What is the relation between $\mathcal{M}$ and $\mathcal{M}_W$? It is remarkable fact, which is discussed in more detail in appendix A, that the two moduli spaces are isomorphic:[7]

$$\mathcal{M}_W \overset{\text{locally}}{\approx} \mathcal{M} \,. \tag{3.22}$$

[7]More precisely, the two deformation functors are equivalent.

This follows from the observation [22, 17, 11, 10] that the algebras $(K, \{r_n\}_{n\geq 3})$ and $(\mathcal{H}, Q, \bullet)$ are *quasi-isomorphic* as A_∞ algebras, i.e. their products are related by a sequence of maps which satisfy certain constraints and whose first element induces an isomorphism between K and the BRST cohomology of $\mathcal{H}$ (the precise definition is recalled in appendix A). In fact, the homotopy algebra $(K, \{r_n\}_{n\geq 2})$ is a so-called *minimal model* [27] for $(\mathcal{H}, Q, \bullet)$, if the later is viewed as an A_∞ algebra whose third and higher products vanish.

The quasi-isomorphism in question is defined by a sequence of maps $F_n : K \to \mathcal{H}$ obtained upon replacing P with U in the definition (2.26) of the string products r_n:

$$r_n(u_1 \dots u_n) = U\lambda_n(u_1 \dots u_n)\,, \tag{3.23}$$

for $u_1 \dots u_n \in K$. This defines F_n for $n \geq 2$. One also needs a map $F_1 : K \to \mathcal{H}$, which we take to be the inclusion (this induces an isomorphism between K and $H_Q(\mathcal{H})$ by Hodge theory, which is why we obtain a quasi-isomorphism). This explicit construction of F is due to [10].

Since F gives a quasi-isomorphism of A_∞ algebras, it is reasonable to expect that it also gives a quasi-isomorphism of L_∞ algebras between their commutator algebras $(K, \{m_n\})$ and $(\mathcal{H}, Q, [.,.])$. This somewhat elementary statement is proved in appendix A.[8]

The final step is to recall from [9] that the so-called deformation functors of two quasi-isomorphic L_∞ algebras are equivalent, which implies that the associated moduli spaces are isomorphic. In our case, the isomorphism follows by noticing that the map:

$$\phi \to F_*(\phi) = \sum_{n\geq 1} F_n(\phi^{\otimes n}) \tag{3.24}$$

takes solutions of the extremum equations (3.17) into solutions of the string field equations of motion (3.1). The inverse correspondence follows from the general result [9] that a quasi-isomorphism of L_∞ algebras always admits a quasi-inverse, i.e. there exists an L_∞ quasi-isomorphism $G : \{g_n : \mathcal{H}^n \to K\}_{n\geq 1}$ such that G_1 induces the inverse isomorphism $(G_1)^* = (F_1^*)^{-1}$ between $H_Q(\mathcal{H})$ and K. Once such a quasi-inverse has been chosen, one obtains a map:

$$\phi \to G_*(\phi) = \sum_{n\geq 1} G_n(\phi^{\otimes n}) \tag{3.25}$$

which takes solutions of (3.1) into solutions of (3.17). Upon combining these two facts, it is not hard to prove the desired equivalence of deformation functors [9]. We will have no need for this inverse correspondence, so we shall omit its explicit realization.

[8]The statement is undoubtedly known to experts in homotopy algebras, but I include a proof since I could not find a convenient reference.

It follows that *one can compute the moduli space of a cubic string field theory either by solving the string field equations of motion, or by extremizing the potential W, and the two results are locally assured to coincide.* Which of these two points of view one chooses depends on what is more convenient in the problem at hand. The cubic formulation gives the simpler-looking Maurer-Cartan equations (3.1), but requires knowledge of the BRST operator Q. The gauge-fixed formulation does not require this datum, but involves the entire sequence of products r_k.

Observations. 1. The correspondence (3.24) mixes the order of deformations. For example, if we have a solution $\phi = \sum_i t_i \phi_i$ to (3.17) in some deformation parameters t_i (where ϕ_i form a basis of K^1), then the corresponding solution $F_*(\phi)$ of (3.1) involves higher orders in t_i. In terms of the associated moduli spaces, this means that (versal) solutions to (3.17) and (3.1) describe local coordinate systems on $\mathcal{M} \approx \mathcal{M}_W$ which differ by a change of coordinates given by a power series.

2. Our explicit description of the potential gives a general method for computing this quantity. This description agrees manifestly with string perturbation theory.

3. Our potential depends on the choice of metric h which enters the gauge-fixing procedure. It is clear that a change $h \to h'$ of this metric induces a quasi-isomorphism between the resulting A_∞ algebras $(K, \{r_n\})$ and $(K', \{r'_n\})$ of string products. By the same argument as above, this implies that the resulting moduli spaces $\mathcal{M}_W$ and $\mathcal{M}'_W$ are isomorphic. Hence a change in the choice of metric corresponds to change of coordinates on $\mathcal{M}$. The associated transition functions will generally involve power series.

4. That the two descriptions of the moduli space agree was expected based on the physical interpretation of W as a tree-level potential for the physical modes. The fact that this intuitive interpretation is strictly correct is, however, entirely nontrivial. As we saw above, its proof makes heavy use of results in modern deformation theory.

5. Our construction gives a string-field theoretic explanation for the appearance of homotopy algebras in *cubic* string field theory. Its application to the topological A and B models (to be discussed below) gives one reason for the relevance of such structures in homological mirror symmetry [8]. It is a general principle of modern topology and deformation theory that many problems can be better understood by enlarging the class of differential graded (associative, Lie...) algebras to the class of their homotopy versions. The double description of moduli spaces discussed above gives an explicit example of the relevance of this principle to string field theory. The fact that homotopy structures play a fundamental role in string theory can be traced back to its relation with loop spaces [43].[9] It is clear for many reasons that a deeper understanding of string theory requires systematic use of this language. For work in this direction I refer the reader to the basic references [40, 21], as well as to the more recent papers [35, 36, 24, 44, 45].

[9]I am grateful to D. Sullivan for an illuminating discussion of these issues.

3.4 An intrinsic formulation of D-brane moduli spaces

In physical applications, cubic string field theory arises as an off-shell description of the dynamics of open strings whose endpoints lie on a D-brane (figure 3). (This includes the case of purely von Neumann boundary conditions). In this situation, one is interested in giving a string-theoretic description of the associated D-brane moduli space.

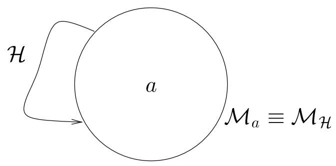

Figure 3: We define the moduli space of a D-brane a to be the moduli space of the open string field theory of strings stretching from a to a. The latter is defined on the off-shell state space $\mathcal{H}$ of such strings.

When studying such moduli spaces, the prevalent procedure has been to approach the problem either from a space-time, geometric point of view, or from a sigma model or conformal field theory perspective. In the first case, one identifies D-brane moduli with the moduli of some geometric object, and then attempts to compute its 'stringy corrections' by considering some auxiliary construction (such as the partially-wrapped D-branes of [1, 2]). The second approach [28] relies on a study of marginal deformations of a boundary conformal field theory [29].

I would like to propose a different perspective on this issue, which attacks the problem via the methods of string field theory. Namely, *we define deformations of the D-brane a to be the vacuum deformations of the string field theory of open strings whose endpoints end on a.* If the latter admits a cubic formulation, then the resulting moduli space can be described in terms of the Maurer-Cartan equation (3.1).[10] It then follows from our results that the same space can be described in terms of the extremum equations (3.17) for the tree level potential W. This gives an alternate formulation of the same problem, which is equivalent with the string field theory approach, and recovers some of the conformal field theory perspective in a computationally efficient manner. In the next section, we apply this method to the simple case of open topological A/B strings compactified on a Calabi-Yau threefold.

4. Application to topological A/B models

4.1 The B model

4.1.1 The geometric data

We consider a Calabi-Yau threefold X and a B-type brane described by a holomorphic vector bundle E over X. The string field theory of strings whose endpoints lie on this brane is the (open) holomorphic Chern-Simons theory of [25]. This has the off-shell state space $\mathcal{H} = \Omega^{0,*}(E^* \otimes E)$, the associative product $\bullet = \wedge$ given by wedge product of bundle-valued forms (this includes composition of bundle morphisms),and

[10] This perspective is the starting point of the papers [3, 4] which gave a general analysis of moduli spaces and condensation processes in a system containing an arbitrary collection of D-branes.

the BRST operator $Q_E = \overline{\partial}$ (the Dolbeault differential coupled to the bundle E). The worldsheet degree is given by form rank ($|u| = p$ if $u \in \Omega^{0,p}(E^* \otimes E)$) and the bilinear form on $\mathcal{H}$ is given by:

$$\langle u, v\rangle = \int_X \Omega \wedge \mathrm{tr}_E(u \wedge v)\,, \tag{4.1}$$

where tr_E denotes the fiberwise trace on the bundle $\mathrm{End}(E) = E^* \otimes E$.

In order to obtain a perturbative expansion, one must choose a gauge and define propagators. To this end, we pick a hermitean metric g_E on E. Together with the Calabi-Yau metric g on X, this induces a metric $(.,.)_E$ on the bundle $\Lambda^*(T^*X \oplus \overline{T}^*X) \otimes \mathrm{End}(E)$. If $u = \omega \otimes \alpha$ and $v = \eta \otimes \beta$ are decomposable elements of $\Omega^{*,*}(\mathrm{End}(E))$, then:

$$(u, v)_E = (\omega, \eta)\, \mathrm{tr}_E(\alpha^\dagger \circ \beta)\,, \tag{4.2}$$

where (ω, η) is the metric induced by g on $\Lambda^*(T^*X \oplus \overline{T}^*X)$, normalized as in [46]. Note that we take all hermitean metrics to be antilinear with respect to the *first* variable. The metrics $(.,.)$ and $(.,.)_E$ allow one to define antilinear Hodge operators $\overline{*}$ and $\overline{*}_E$ on $\Omega^{*,*}(X)$ and $\Omega^{*,*}(E^* \otimes E)$, which take $\Omega^{p,q}(X)$ into $\Omega^{3-p,3-q}(X)$ and $\Omega^{p,q}(E^* \otimes E)$ into $\Omega^{3-p,3-q}(E^* \otimes E)$. The operator $\overline{*}_E$ is the tensor product of $\overline{*}$ with the hermitean conjugation $\dagger$ in $\mathrm{End}(E)$, taken with respect to the metric g_E:

$$\overline{*}_E(\omega \otimes \alpha) = \overline{*}\omega \otimes \alpha^\dagger\,. \tag{4.3}$$

With these conventions, one has the relations:

$$\overline{*}1_X = \mathrm{vol}_g\,, \qquad (\overline{*}\omega) \wedge \eta = (\omega, \eta)\,\mathrm{vol}_g\,, \qquad \mathrm{tr}_E[(\overline{*}_E u) \wedge v] = (u, v)_E\,\mathrm{vol}_g\,, \tag{4.4}$$

where 1_X is the unit function defined on X, 1_E is the identity section of $\mathrm{End}(E)$ and vol_g is the volume form induced by the Calabi-Yau metric g. We recall that $\overline{*}_E$ satisfies:

$$\overline{*}_E^2 u = (-1)^{rku} u\,, \tag{4.5}$$

as a consequence of the similar property of $\overline{*}$.

4.1.2 The operator c

We next define a hermitean product h on $\mathcal{H}$ through the standard relation:

$$h(u, v) = \int_X \mathrm{tr}_E[(\overline{*}_E u) \wedge v] = \int_X (u, v)_E\,\mathrm{vol}_g\,. \tag{4.6}$$

where $u, v \in \Omega^{0,*}(E^* \otimes E)$. This scalar product vanishes on homogeneous elements u, v unless their ranks coincide. The relation between $\overline{*}_E$ and h is similar to the relation between c and h discussed in section 2, but with respect to the 'wrong' bilinear form $\xi(u, v) = \int_X \mathrm{tr}_E(u \wedge v)$ (rather than the physically correct form (4.1)).

In fact, the Hodge operator $\bar{*}_E$ does not preserve the space $\mathcal{H}$, and therefore it cannot be identified with the conjugation c. Indeed, $\bar{*}$ maps $\Omega^{0,q}(E^*\otimes E)$ into $\Omega^{3,3-q}(E^*\otimes E)$, which is not a subspace of $\mathcal{H}$. Following the discussion of section 2, we define an operator $c : \Omega^{0,*}(E^* \otimes E) \to \Omega^{0,3-*}(E^* \otimes E)$ through the relation $h(u, v) = \langle cu, v\rangle$, i.e.:

$$\int_X \mathrm{tr}_E\left[(\bar{*}_E u) \wedge v\right] = \int_X \Omega \wedge \mathrm{tr}_E(cu \wedge v)\,. \tag{4.7}$$

Since this must hold for all v, we take:

$$\bar{*}_E u = \Omega \wedge cu\,, \tag{4.8}$$

which uniquely determines the antilinear map c. In order to satisfy the framework of section 2, we must check that c squares to the identity. We show that this can be fulfilled by normalizing the holomorphic 3-form through:

$$(\bar{*}\Omega) \wedge \Omega = \mathrm{vol}_g\,, \tag{4.9}$$

where vol_g is the volume form induced by the Calabi-Yau metric g. This normalization condition can always be satisfied by a constant rescaling of Ω. The proof of the identity $c^2 = id$ proceeds in three steps:

1. We first show the relation:

   $$\mathrm{tr}_E(c^2u \wedge v) = \mathrm{tr}_E(u \wedge c^2v)\,,\ \text{ for } u, v \in \mathcal{H} = \Omega^{0,*}(\mathrm{End}(E))\,. \tag{4.10}$$

   For this, notice that the last of equations (4.4) and the definition (4.8) imply:

   $$\Omega \wedge \mathrm{tr}_E(cu \wedge v) = (u, v)_E\, \mathrm{vol}_g\ . \tag{4.11}$$

   Upon permuting u and v in this equation and using the hermicity of $(.,.)_E$, one obtains:

   $$\Omega \wedge \mathrm{tr}_E(cu \wedge v) = \overline{\Omega \wedge \mathrm{tr}_E(cv \wedge u)}\,. \tag{4.12}$$

   Relation (4.10) now follows by repeated application of (4.12), combined with the graded symmetry of the wedge product.

2. We next compute the value of $c^2(\alpha)$ for α a local section of $\mathrm{End}(E)$. According to (4.8), the $(0, 3)$ form $c(\alpha)$ is determined by:

   $$\bar{*}_E\alpha = \Omega \wedge c(\alpha)\,. \tag{4.13}$$

   On the other hand, the first relation in (4.4) implies $\bar{*}_E\alpha = \mathrm{vol}_g \otimes \alpha^\dagger$. Combining this with (4.13) and (4.9) gives:

   $$c(\alpha) = -\bar{*}\Omega \otimes \alpha^\dagger = -\bar{*}_E(\Omega \otimes \alpha)\,. \tag{4.14}$$

We next determine the 0-form $c^2(\alpha) = -c(\bar{*}\Omega \otimes \alpha^\dagger)$ from the defining relation (4.8):

$$\bar{*}_E(\bar{*}\Omega\otimes\alpha^\dagger) = \Omega\wedge c(\bar{*}\Omega\otimes\alpha^\dagger) \Leftrightarrow \Omega\otimes\alpha = -\Omega\otimes c(\bar{*}\Omega\otimes\alpha^\dagger) \Leftrightarrow -c(\bar{*}\Omega\otimes\alpha^\dagger) = \alpha\,, \tag{4.15}$$

where we used $\bar{*}_E(\bar{*}\Omega \otimes \alpha^\dagger) = \bar{*}_E^2(\Omega \otimes \alpha) = -\Omega \otimes \alpha$, due to property (4.5) of the Hodge operator. Equation (4.15) shows that:

$$c^2(\alpha) = \alpha\,. \tag{4.16}$$

This nice relation is a consequence of the normalization condition (4.9).

3. We are now ready to show that $c^2 = Id_{\mathcal{H}}$. For this, we take $v = \alpha$ in relation (4.10) and use the property (4.16) to obtain:

$$\mathrm{tr}_E[(c^2 u)\alpha] = \mathrm{tr}_E(u\alpha)\,. \tag{4.17}$$

Since α is an arbitrary local section of $\mathrm{End}(E)$, this leads to the desired conclusion:

$$c^2(u) = u \qquad \text{for } u \in \mathcal{H} = \Omega^{0,*}(\mathrm{End}(E))\,. \tag{4.18}$$

Hence all arguments of sections 2 and 3 apply.

4.1.3 The potential

Proceeding as in section 2, we pick the gauge:[11]

$$Q_E^+ u = 0 \qquad \text{for } u \in \Omega^{0,*}(\mathrm{End}(E))\,, \tag{4.19}$$

where $Q_E^+ = \overline{\partial}^+ = -\bar{*}\overline{\partial}*$ is the hermitean conjugate of $Q_E = \overline{\partial}$ with respect to the scalar product (4.6) (this can also be expressed in the form (2.12)). The associated hamiltonian $H = QQ^+ + Q^+Q$ is the $\overline{\partial}$-laplacian:

$$H = \Delta_{\overline{\partial}} = \overline{\partial}^+\overline{\partial} + \overline{\partial}\overline{\partial}^+\,. \tag{4.20}$$

The BRST cohomology is then represented by physical states, i.e. degree one states lying in the kernel $K_E = \ker Q_E \cap \ker Q_E^+$; these are just the harmonic forms in $\Omega^{0,1}(\mathrm{End}(E))$. The operator P of section 2 is the orthogonal projector on the space K of harmonic $(0,*)$-forms, while the propagator U has the form:

$$U = \overline{\partial}^+ G = \frac{1}{\Delta_{\overline{\partial}}}\overline{\partial}^+\,, \tag{4.21}$$

where $G = \frac{1}{\Delta_{\overline{\partial}}}(1-P)$ is a Green's function for $\Delta_{\overline{\partial}}$.

[11] This is an analogue of the Siegel gauge of bosonic string theory. Indeed, the analogue of the bosonic antighost operator b_0 in a topological string theory obtained by twisting an $N = 2$ superconformal field theory is the generator $\mathcal{G}_0$ of the topological ('twisted') $N = 2$ algebra. In a unitary conformal field theory, this is the hermitean conjugate of the generator $\mathcal{Q}_0$, which can be identified with the BRST charge. In our treatment, this is represented in spacetime through the Dolbeault differential $\overline{\partial}$ via the localization argument of [25].

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As in section 2, one can express the disk string correlators $\langle\langle u_0 \dots u_n \rangle\rangle^{(n)}$ of $n+1$ physical states in terms of tree level Feynman rules. The resulting potential has the form:

$$W(\phi) = \sum_{n \geq 2} \frac{(-1)^{n(n+1)/2}}{n+1} \int_X \Omega \wedge \mathrm{tr}_E(\phi \wedge r_n(\phi^{\otimes n})) \,, \tag{4.22}$$

with $\phi \in \Omega^{0,1}_{\overline{\partial}-\mathrm{harm}}(\mathrm{End}(E))$ and the products r_n defined as explained there. It is easy to see that these coincide with the products considered in [17], were they were introduced without a physical justification. As explained there, r_n induce the holomorphic version of Massey products on $H^{0,*}_{\overline{\partial}}(\mathrm{End}(E)) \approx H^*_{\mathrm{sheaf}}(\mathrm{End}(E))$, when the latter are well-defined and one-valued. Unlike the Massey products, though, the string products r_n are *always* well-defined; hence they give an extension of classical Massey theory.

The argument of section 2 implies that r_n have the cyclicity properties (2.32) *with respect to the topological metric* (4.1). I wish to caution the reader that this is *not* the bilinear form used in [17]. Indeed, the paper cited proves cyclicity with respect to the 'wrong' bilinear form $\xi(u,v) = \int_X \mathrm{tr}_E(u \wedge v)$, which is related to Serre duality. As explained above, this does not coincide with the bilinear form relevant for B-model physics. The physically relevant object is the topological metric (4.1), and our construction of the operator c, together with the general arguments of section 2, show that the cyclicity result of [17] remains valid with respect to this form.

It is clear that $W(\phi)$ corresponds to a particular case of the 'D-brane superpotentials' considered in [1, 2]. In our situation, this interpretation arises upon considering some 9-brane partially wrapped over X and filling the uncompactified spacetime directions. As mentioned above, this point of view is not intrinsic and is affected by difficulties due to flux conservation (which must be cured by restricting to non-compact Calabi-Yau manifolds or positing some orientifold construction). For these reasons, we prefer the string field interpretation.

4.1.4 The moduli space

As outlined above, it is natural to define the moduli space $\mathcal{M}_E$ of our B-type brane as the moduli space of vacua of the associated string field theory. As discussed in section 3, this admits two equivalent presentations:

String field theory description. In this approach, $\mathcal{M}_E$ is obtained by solving the Maurer-Cartan equations:

$$\overline{\partial}\phi + \phi \wedge \phi = 0 \tag{4.23}$$

and dividing through the gauge group action generated by:

$$\phi \to \phi - \overline{\partial}\alpha - [\phi, \alpha] \,, \tag{4.24}$$

with $\alpha \in \Omega^{0,0}(\mathrm{End}(E)) = \Gamma(\mathrm{End}(E))$ a (smooth, i.e. $\mathcal{C}^\infty$) section of $\mathrm{End}(E)$. The Maurer-Cartan equation is equivalent with $(\overline{\partial} + \phi)^2 = 0$, which is the condition that $\overline{\partial} + \phi$ determines an integrable complex structure; the gauge transformations identify equivalent complex structures. The string field theory gauge group $\mathcal{G}$ can be identified with the group $\mathrm{Aut}(E)$ of smooth ($\mathcal{C}^\infty$) automorphisms λ of E, whose adjoint action on $\mathcal{H} = \Omega^{0,*}(\mathrm{End}(E))$ has the form:

$$u \to \lambda \circ u \circ \lambda^{-1} \,. \tag{4.25}$$

Finite gauge transformations of the string field $\phi \in \mathcal{H}^1 = \Omega^{0,1}(\mathrm{End}(E))$ are given by:

$$\phi \to \lambda \circ \overline{\partial}\lambda^{-1} + \lambda \circ \phi \circ \lambda^{-1} \,. \tag{4.26}$$

Therefore, $\mathcal{M}_E$ is the moduli space of complex structures on E (more precisely, its component which contains the original complex structure $\overline{\partial}$). Note that both the space of integrable connections and the gauge group are infinite dimensional, so we have an 'infinite presentation' of the finite-dimensional space $\mathcal{M}_E$, i.e. a presentation as the quotient of an infinite-dimensional space through the action of an infinite dimensional transformation group.

Description trough the D-brane superpotential. According to our general arguments, the same moduli space can be described in terms of solutions to the 'F-flatness' condition $\delta W/\delta\phi = 0$, which leads to equation (3.8). To identify the symmetry group G_W, notice that K^0 is the space $\Gamma_{\mathrm{hol}}(\mathrm{End}(E))$ of (global) holomorphic sections of $\mathrm{End}(E)$. It is clear that $\overline{\partial}^+(\alpha u) = \alpha\overline{\partial}^+ u$ for any holomorphic section α of $\mathrm{End}(E)$ and any $\mathrm{End}(E)$-valued $(0,q)$-form u; thus hypothesis (3.19) of subsection 3.3. is satisfied. The Lie algebra of G_W is $(K^0, [.,.])$, which is simply the Lie algebra $\Gamma_{\mathrm{hol}}(\mathrm{End}(E))$ of (global) holomorphic sections of $\mathrm{End}(E)$, while the effective symmetry group is the group $\mathrm{Aut}_{\mathrm{hol}}(E)$ of holomorphic automorphisms of E. This acts on $\mathcal{H}^1 = \Omega^{0,1}(\mathrm{End}(E))$ in its adjoint representation:

$$\phi \to \lambda\phi\lambda^{-1} \,, \tag{4.27}$$

for $\phi \in \Omega^{0,1}(\mathrm{End}(E))$ and $\lambda \in \mathrm{Aut}_{\mathrm{hol}}(E)$. This action preserves the space K^1 of $\mathrm{End}(E)$-valued harmonic $(0,1)$-forms, on which it restricts to the action (3.21) of subsection 3.3. Hence the moduli space of complex structures on E can be locally described as the space of solutions $\phi \in K^1 = \Omega^{0,1}_{\overline{\partial}-\mathrm{harm}}(\mathrm{End}(E))$ of the 'F-flatness conditions' $\delta W/\delta\phi = 0$, modded out by the group action (4.27). This gives the precise realization for our system of the proposals of [1] and [2]. The result is a 'finite presentation' of $\mathcal{M}_E$, as the quotient of a finite-dimensional space through the action of a finite-dimensional group.

Observation. A thorough analysis of the two presentations involves functional analytic, respectively complex-analytic/algebro-geometric issues. How to deal with these is a standard subject. Our description of the moduli problems is in fact entirely formal (even locally), since we did not address the problem of building a complex analytic structure on the moduli space (which requires Kuranishi theory). Strictly speaking, we are only studying the associated deformation functors, rather than the moduli spaces themselves.

4.2 The A model

The A model realization is also easy to obtain (part of this is already sketched in [25]). For this, we consider an A-type brane described by a pair (L, E) with L a special lagrangian cycle of a Calabi-Yau threefold X and E a flat complex vector bundle over E. We remind the reader that a flat structure on E can be described either in terms of a collection of local trivializations whose transition functions are constant, or in terms of a flat connection A on E. The latter defines a differential d on $\mathrm{End}(E)$-valued forms, the Dolbeault differential 'twisted by A'. The string field theory in our boundary sector is described by the data $\mathcal{H} = \Omega^*(E)$, $Q = d$, $\bullet = \wedge$ and:

$$\langle u, v\rangle = \int_L \mathrm{tr}_E(u \wedge v) \qquad \text{for } u, v \in \Omega^*(L, \mathrm{End}(E))\,. \tag{4.28}$$

This is the Chern-Simons field theory on E:

$$S(\phi) = \int_L \mathrm{tr}_E\left[\frac{1}{2}\phi \wedge d\phi + \frac{1}{3}\phi\wedge\phi\wedge\phi\right]. \tag{4.29}$$

The perturbation expansion is obtained by picking a hermitean metric on E. If $*$ is the associated Hodge operator, we have an induced hermitean product on $\mathcal{H}$ given by:

$$h(u, v) = \int_L \mathrm{tr}_E(*u \wedge v)\,. \tag{4.30}$$

The hermitean conjugate of $Q = d$ with respect to this product is $Q^+ = d^+ = - * d *$, and the associated hamiltonian is the d-laplacian, $H = \Delta = dd^+ + d^+d$ (coupled to the flat bundle E). The space K^1 of physical states consists of d-harmonic $\mathrm{End}(E)$-valued one-forms on L. In this case, one has $*^2 = Id$ and the antilinear map c of section 3 is simply the Hodge operator $*$. The string field theory gauge group $\mathcal{G}$ is the group of $\mathcal{C}^\infty$ automorphisms of E, acting on string fields $\phi \in \mathcal{H}^1 = \Omega^1(\mathrm{End}(E))$ through:

$$\phi \to \lambda \circ \phi \circ \lambda^{-1} + \lambda \circ d\lambda^{-1}\,. \tag{4.31}$$

It is clear that the Maurer-Cartan equations (3.1) give the moduli space $\mathcal{M}_E$ of flat connections on E. This can also be described in terms of extrema of the potential:

$$W(\phi) = \sum_{n\geq 2} \frac{(-1)^{n(n+1)/2}}{n+1}\int_L \mathrm{tr}_E(\phi \wedge r_n(\phi^{\otimes n}))\,, \tag{4.32}$$

with the string products r_n built as in section 3. The effective symmetry group G_W is the group $\mathrm{Aut}_{\mathrm{flat}}(E)$ of covariantly constant gauge transformations (automorphisms of E as a *flat* vector bundle). This acts on elements of $K^1 = \Omega^1_{harm}(\mathrm{End}(E))$ through its adjoint representation. We obtain a finite presentation of the moduli space of flat connections on E.

Since we wish to obtain a *complex* moduli space, *we do not require the one-forms ϕ to be anti-hermitean.* This amounts to considering flat connections on E which are not subject to a hermicity condition (hence the Chern-Simons action becomes complex-valued, as was the case for the B-model; alternately, one can replace this complex action through its imaginary part etc). Similarly, elements of the groups $\mathcal{G}$ and G_W are not required to be unitary. The (somewhat non-rigorous) justification for this is that the moduli space of flat anti-hermitean connections describes only half of A-brane moduli. The other half, which is due to deformations of the special lagrangian cycle, is expected to pair up with the former due to $N = 1$ space-time supersymmetry – a fact which is locally described by lifting the antihermicity condition. This can be seen explicitly for the case of a singly-wrapped brane (i.e. $\mathrm{rank}\, E = 1$), upon using the results of [31] (see [32] for a nice exposition). For multiply-wrapped branes ($\mathrm{rank}\, E > 1$), the situation is less clear, though it seems [33] that one could study that case by viewing such objects as degenerations of singly-wrapped branes. The procedure of lifting the antihermicity constraint should be trusted only locally – the global structure of the moduli space is clearly much more involved. A thorough physical analysis of this issue seems to require an extension of the open A-model of [25], which should be obtained by considering supplementary boundary couplings (couplings to sections of the normal bundle to the special lagrangian cycle L, which describe its deformations). This sort of extension does not seem to have been studied, despite its relevance for mirror symmetry and Calabi-Yau D-brane physics.[12] An analysis of such a generalized model would presumably make contact with the mathematical construction of [34].

5. Conclusions and directions for further research

We presented a detailed analysis of tree level boundary potentials in (cubic) open string field theory, giving a general prescription for their construction in terms of the

[12]The results of [31] assure that deformations of a special lagrangian cycle L can be represented through one-forms on the cycle. In the presence of a multiply-wrapped brane on L, these do not pair up with deformations of the flat connection in any obvious manner. One way out is to view a multiply wrapped brane as involving n copies of the cycle L lying on top of each other, and construct the higher rank Chan-Paton bundle E on L from a collection of line bundles, each living on one of the copies of L. This is complicated by the fact that one generally must also take the B-field into account. It is unclear if such a construction allows one to recover the most general physical situation. Moreover, its significance is doubtful unless one can show equivalence with a generalized A model which explicitly couples to deformations of L.

associated string products. By analyzing the resulting moduli problem, we gave its formulation in terms of modern deformation theory and *proved* the equivalence of the string field and 'low energy' descriptions of the moduli space. The proof makes use of recent results in the theory of homotopy algebras. Upon applying these methods to the topological A and B models, we identified our potentials with the D-brane superpotentials of [1, 2] and gave their precise description in terms of geometric quantities. This clarifies the meaning of these objects and explains the appearance of homotopy associative and homotopy Lie algebras in the *associative* open string field theories of the B-model and in the large radius limit of the A-model, thereby establishing part of the connection with mathematical work on homological mirror symmetry. The fact that this connection appears naturally from the string field point of view brings further evidence in favor of the program outlined in [4] of recovering (and potentially extending) homological mirror symmetry by means of string field theory.

The fact that deformations of flat and holomorphic vector bundles are potential in this sense is a direct consequence of the existence of a string field theory description. This can be formulated more generally in deformation-theoretic terms, by introducing a bilinear form and appropriate 'conjugation' operator c within the framework of classical (Maurer-Cartan) or generalized (homotopy Maurer-Cartan) deformation theory (with a non-vanishing first order product). The first situation corresponds to cubic (or associative) open string field theory [25], while the second is described by the non-polynomial (homotopy associative) open string field theory of [35, 36]. The arguments of the present paper can be generalized to the second case, which implies the existence of a potential for a variety of deformation problems important in physics. For example, it seems that this approach can shed light on the existence of a potential for holomorphic curves embedded in a Calabi-Yau threefold, thereby explaining and generalizing some results of [2, 37]. We hope to return this issue in future work.

Another extension involves the inclusion of instanton corrections in the A model. One reason for restricting to the large radius limit is the fact that disk instantons are responsible for a series of effects which require a rather sophisticated analysis. A precise discussion can be given with string field theory methods, and makes contact with the work of K. Fukaya [14, 15].

Finally, it is important to extend the analysis of the present paper by including interactions between open and closed strings. This can be carried out by considering the tree-level restriction of the string field theory of [36], which is governed by a so-called G_∞ (homotopy Gerstenhaber) algebra and provides the off-shell extension of the framework of [38]. The associated moduli space describes joint deformations of the Calabi-Yau manifold X and a D-brane, for example simultaneous deformations of the complex structure of X and of a holomorphic vector bundle E on X (for the case of the B-model). The string field approach should lead to an explicit description of

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the relevant potentials, as well a detailed analysis of the deformation problem (and, for the A-model, of its 'quantization'). The infinitesimal version of such deformations was analyzed to first order in [24]. As in the open string case, string field theory can provide a more explicit construction.

Acknowledgments

I wish to thank S. Popescu for collaboration in a related project and D. Sullivan and E. Diaconescu for useful conversations. I also thank R. Roiban for comments on the manuscript. I am endebted to M. Rocek for constant support and interest in my work. The author is supported by the Research Foundation under NSF grant PHY-9722101.

A. Homotopy algebras and their morphisms

This appendix summarizes some basic facts about (strong) homotopy associative and homotopy Lie algebras, and gives a proof of the statement that an A_∞ quasi-isomorphism induces an L_∞ quasi-isomorphism between the commutator algebras. Most of the appendix is written in an explicit, 'component' language. However, the proof itself makes use of a dual description in terms of codifferential coalgebras, since this avoids computational morass. For this, I shall assume that the reader is familiar with the theory of coalgebras. This appendix is intended for the convenience of non-expert readers. Readers familiar with the subject may wish to skip to appendix B.

A.1 A_∞ algebras

For more information on this subject the reader is referred to [12, 26, 27, 42, 39] and the references therein. A basic reference discussing the role of A_∞ algebras in open string field theory is [35].

Definition. A (strict, or strong) A_∞ algebra is defined by a $\mathbf{Z}$-graded vector space A together with a countable collection of operations $r_n : A^{\otimes n} \to A$ $(n \geq 1)$ which are homogeneous of degree $2-n$ and subject to the constraints:

$$\sum_{\substack{k+l=n+1\\ j=0\dots k-1}} (-1)^s r_k(u_1 \dots u_j, r_l(u_{j+1} \dots u_{j+l}), u_{j+l+1} \dots u_n) = 0\,, \tag{A.1}$$

for $n \geq 1$, where $s = l(|u_1| + \dots |u_j|) + j(l-1) + (k-1)l$ and $|\,.\,|$ denotes the degree of homogeneous elements in A.

Observation. A *weak* A_∞ algebra [14, 15] is defined in a similar manner, but it also contains a 0^{th} order product r_0. We shall have no use for weak A_∞ algebras in this paper. Such structures are relevant for describing open string field theory built around a background which does not satisfy the string equations of motion.

The first three constraints (A.1) read:

$$\begin{aligned} r_1^2 &= 0 \\ r_1(r_2(u_1,u_2)) &= r_2(r_1(u_1),u_2) + (-1)^{|u_1|} r_2(u_1, r_1(u_2)) \\ r_2(u_1, r_2(u_2,u_3)) - r_2(r_2(u_1,u_2),u_3) &= \\ &= r_1(r_3(u_1,u_2,u_3)) + r_3(r_1(u_1),u_2,u_3) + (-1)^{|u_1|} r_3(u_1, r_1(u_2), u_3) + \\ &\quad + (-1)^{|u_1|+|u_2|} r_3(u_1,u_2,r_1(u_3))\,. \end{aligned} \tag{A.2}$$

In particular, r_1 is a degree one differential on A. In the case $r_n = 0$ for $n \geq 3$, an A_∞ algebra reduces to a differential graded associative algebra with differential $Q = m_1$ and product $u \bullet v = r_2(u,v)$.

Definition 1. *Given a (strict) A_∞ algebra $(A, \{r_n\}_{n\geq 1})$, its* cohomology $H_{r_1}(A)$ *is the cohomology of the vector space A with respect to the differential r_1.*

Definition 2. [39] *Given two (strict) A_∞ algebras $(A, \{r_n\}_{n\geq 1})$ and $(A', \{r'_n\}_{n\geq 1})$, an A_∞* morphism *$f = \{f_n\}_{n\geq 1}$ from A to A' is a collection of maps $f_n : A^{\otimes n} \to A'$ which are homogeneous of degree $1-n$ and satisfy the conditions:*

$$\begin{aligned} &\sum_{1\leq k_1 < k_1 \ldots < k_i = n} (-1)^r r'_i(f_{k_1}(u_1 \ldots u_{k_1}), f_{k_2-k_1}(u_{k_1+1} \ldots u_{k_2}) \ldots f_{n-k_{i-1}}(u_{k_{i-1}+1} \ldots u_n)) = \\ &= \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{|u_1|+\ldots+|u_j|+j+s} f_k(u_1 \ldots u_j, r_l(u_{j+1} \ldots u_{j+l}), u_{j+l+1} \ldots u_n)\,, \end{aligned} \tag{A.3}$$

for $n \geq 1$. The exponents r and s in these relations are given by:

$$\begin{aligned} r &= \mu(u_1, \ldots, u_{k_1}) + \mu(u_{k_1+1}, \ldots, u_{k_2}) + \ldots + \mu(u_{k_{i-1}+1}, \ldots, u_n) + \\ &\quad \mu(f_{k_1}(u_1, \ldots, u_{k_1}), \ldots, f_{n-k_{i-1}}(u_{k_{i-1}+1}, \ldots, u_n))\,, \\ s &= \mu(u_{j+1}, \ldots, u_{j+l}) + \mu(u_1, \ldots, u_j, r_l(u_{j+1}, \ldots, u_{j+l}), u_{j+l+1}, \ldots, u_n)\,, \end{aligned} \tag{A.4}$$

where

$$\mu(a_1, \ldots, a_k) := (k-1)|a_1| + (k-2)|a_2| + \ldots + |a_{k-1}| + \frac{k(k-1)}{2}\,. \tag{A.5}$$

The first two constraints in (A.3) read:

$$\begin{aligned} r'_1(f_1(u)) &= f_1(r_1(u)) \\ r'_2(f_1(u_1), f_1(u_2)) &= f_1(r_2(u_1,u_2)) + r'_1(f_2(u_1,u_2)) + \\ &\quad + f_2(r_1(u_1),u_2) + (-1)^{|u_1|} f_2(u_1, r_1(u_2))\,. \end{aligned} \tag{A.6}$$

In particular, f_1 induces a degree zero linear map f_{1*} between the cohomologies $H_{r_1}(A)$ and $H_{r'_1}(A')$.

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Definition 3. *An A_∞ morphism is called a* quasi-isomorphism *if f_{1*} is a degree zero isomorphism between the cohomology spaces $H_{r_1}(A)$ and $H_{r'_1}(A')$. It is a* homotopy *if f_1 is a linear isomorphism (of degree zero) from A to A'. It is clear that any homotopy is a quasi-isomorphism.*

A.1.1 The bar construction

Notation. For any graded vector space V, we let $V[1]$ denote its *suspension*, i.e. the vector space V endowed with the shifted grading $V[1]^k = V^{k+1}$. We let $s : V \to V[1]$ be the identity map of V, viewed as a homogeneous morphism of degree -1.

If V is a graded vector space, then $T(V) = \oplus_{k\geq 1} V^{\otimes k} \subset \otimes^* V$ denotes its reduced tensor coalgebra, with the (coassociative) coproduct $\Delta : T(V) \to T(V) \otimes T(V)$ defined through:

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{j=1}^{n-1} (v_1 \otimes \ldots \otimes v_j) \otimes (v_{j+1} \otimes \ldots \otimes v_n)\,. \tag{A.7}$$

Note that $T(V)$ does not contain the summand $V^{\otimes 0} = \mathbf{C}$. In particular, it does not have a counit.

Proposition 1. *An A_∞ algebra on the graded vector space A is the same as a nilpotent degree one coderivation ∂ on $T(A[1])$, i.e. a homogeneous map of degree one from $T(A[1])$ to itself satisfying the conditions:*

$$\Delta\partial = (Id \otimes \partial + \partial \otimes Id)\Delta \tag{A.8}$$

and:

$$\partial^2 = 0\,. \tag{A.9}$$

If $\pi : T(A[1]) \to A[1]$ denotes projection on the first component, then the product r_n is recovered from the map $\partial_n := \pi \circ \partial|_{A[1]^{\otimes n}}$ by desuspension:

$$r_n = s^{-1} \circ \partial_n \circ (s^{\otimes n})\,. \tag{A.10}$$

Proposition 2. *Given two A_∞ algebras A and A', an A_∞ morphism $f : A \to A'$ is the same as a coalgebra morphism*[13] *$F : T(A[1]) \to T(A'[1])$ which commutes with the codifferential ∂ and ∂' (i.e. a morphism of differential graded coalgebras). The components f_n are obtained by desuspension in the obvious manner.*

[13]This means $\Delta' \circ F = (F \otimes F) \circ \Delta$.

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A.2 L_∞ algebras

More information on this subject can be found in [9, 12, 26] and the references therein. The relevance of L_∞ algebras for (closed) string field theory was discovered in [40] and is explained in detail in [21].

Definition 4. *For a graded vector space V, we let $\odot^* V = \oplus_{k\geq 0} \odot^k V$ denote the associated (graded) symmetric algebra, defined upon dividing the free graded associative algebra $\otimes^* V = \oplus_{k\geq 0} \otimes^k V$ through the homogeneous ideal generated by elements of the form:*

$$u \otimes v - (-1)^{|u||v|} v \otimes u \,, \tag{A.11}$$

where we denote the degree of homogeneous elements u of V by $|u|$. Given a permutation σ on n elements, we define the Koszul sign *$\epsilon(\sigma, u_1 \dots u_n)$ through:*

$$u_{\sigma(1)} \odot \dots \odot u_{\sigma(n)} = \epsilon(\sigma; u_1 \dots u_n) u_1 \odot \dots \odot u_n \,, \tag{A.12}$$

in the algebra $\odot^* V$.

Definition 5. *Given a graded vector space V, we let $\Lambda^* V = \oplus_{k\geq 0} \Lambda^k V$ denote the associated (graded) exterior algebra, defined upon dividing the free graded associative algebra $\otimes^* V$ through the homogeneous ideal generated by elements of the form:*

$$u \otimes v + (-1)^{|u||v|} v \otimes u \,. \tag{A.13}$$

If σ is a permutation on n elements, we define the symbol $\chi(\sigma, u_1 \dots u_n)$ through:

$$u_{\sigma(1)} \wedge \dots \wedge u_{\sigma(n)} = \chi(\sigma; u_1 \dots u_n) u_1 \wedge \dots \wedge u_n \,, \tag{A.14}$$

in the exterior algebra $\Lambda^* V$.

Observation. One has:

$$\chi(\sigma; u_1 \dots u_n) := \epsilon(\sigma) \epsilon(\sigma; u_1 \dots u_n) \,. \tag{A.15}$$

Definition 6. *An L_∞ algebra structure on a graded vector space L is defined by an infinite sequence of products $m_n : \Lambda^n L \to L$ which are homogeneous of degree $2 - n$ and subject to the constraints:*

$$\sum_{k+l=n+1} \sum_{\sigma \in Sh(k,n)} (-1)^{k(l-1)} \chi(\sigma; u_1 \dots u_n) m_l(m_k(u_{\sigma(1)} \dots u_{\sigma(k)}), u_{\sigma(k+1)} \dots u_{\sigma(n)}) = 0 \,, \tag{A.16}$$

for $n \geq 1$.

The first three conditions in (A.16) read:

$$\begin{aligned} m_1^2 &= 0 \\ m_1(m_2(u_1,u_2)) &= m_2(m_1(u_1),u_2) + (-1)^{|u_1|} m_2(u_1, m_1(u_2)) \\ m_2(m_2(u_1,u_2),u_3)) &+ (-1)^{(|u_1|+|u_2|)|u_3|} m_2(m_2(u_3,u_1),u_2) + \\ &\qquad (-1)^{|u_1|(|u_2|+|u_3|)} m_2(m_2(u_2,u_3),u_1) = \\ &= -m_1(m_3(u_1,u_2,u_3)) - m_3(m_1(u_1),u_2,u_3) - \\ &\quad -(-1)^{|u_1|} m_3(u_1,m_1(u_2),u_3) - (-1)^{|u_1|+|u_2|} m_3(u_1,u_2,m_1(u_3)) \,. \end{aligned} \tag{A.17}$$

In particular, the first product m_1 is a degree one differential on L. If the triple and higher products all vanish, then an L_∞ algebra reduces to a differential graded Lie algebra, with differential given by $Q = m_1$ and graded Lie bracket $[u, v] = m_2(u, v)$.

Definition 7. *The cohomology of an L_∞ algebra $(L, \{m_n\}_{n\geq 1})$ is the cohomology $H_{m_1}(L)$ of the complex (L, m_1).*

A.2.1 Dual description and L_∞ morphisms

Notation. Given a graded vector space V, we define its *reduced symmetric algebra* by $S(V) = \oplus_{k\geq 1} \odot^k V$ (this is a subspace of $\odot^* V$). We view $S(V)$ as a (cocommutative, coassociative) coalgebra without counit endowed with the coproduct:

$$\Delta(v_1\odot\ldots\odot v_n) = \sum_{j=1}^{n-1} \sum_{\sigma\in Sh(j,n)} \epsilon(\sigma, v_1\ldots v_n)(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(j)})\otimes(v_{\sigma(j+1)}\odot\cdots\odot v_{\sigma(n)})\,. \tag{A.18}$$

Proposition 3. *An L_∞ algebra on the graded vector space L is the same as a nilpotent degree one coderivation δ on the coalgebra $S(L[1])$, i.e. a degree one linear map from $S(L[1])$ to itself which satisfies:*

$$\Delta\delta = (Id\otimes\delta + \delta\otimes Id)\Delta \tag{A.19}$$

and:

$$\delta^2 = 0\,. \tag{A.20}$$

If $\pi : S(L[1]) \to L[1]$ is the projection on the first factor, then the products m_n are recovered from the maps $\delta_n = \pi\circ\delta|_{L[1]^{\odot n}}$ by desuspension:

$$m_n = s^{-1}\circ\delta_n\circ(s^{\odot n})\,, \tag{A.21}$$

where $s^{\odot n}$ is the map induced from $s^{\otimes n}$ by graded symmetrization.

Definition 8. *An L_∞ morphism between two L_∞ algebras L and L' is specified by a degree zero coalgebra morphism F from $S(L[1])$ to $S(L'[1])$ which commutes with the codifferentials δ and δ'. Its components f_n are defined by desuspension of $\pi'\circ F|_{L[1]^{\odot n}}$. They are homogeneous linear maps of degree $1-n$ from $\Lambda^n L$ to L', which satisfy a countable set of constraints equivalent with the condition $\delta' F = F\delta$.*

The first of these constraints reads:

$$m'_1(f_1(u)) = m_1(r_1(u))\,. \tag{A.22}$$

Hence f_1 induces a degree zero linear map f_{1*} between the cohomologies $H_{m_1}(L)$ and $H_{m'_1}(L')$.

Definition 9. *An L_∞ morphism is a* quasi-isomorphism *if f_{1*} is a (degree zero) isomorphism between the cohomology spaces $H_{m_1}(L)$ and $H_{m'_1}(L')$. It is a* homotopy *if f_1 is an isomorphism of graded vector spaces.*

Observation. Since morphisms of codifferential coalgebras are closed under composition, we have a natural notion of composition of L_∞ morphisms. Hence L_∞ algebras and L_∞ morphisms form a category. An isomorphism in this category is called an L_∞ *isomorphism.* It can be shown [9] that an L_∞-morphism $f : L \to L'$ is an isomorphism if and only if it is a homotopy.

Theorem 1. [9] *Every L_∞ quasi-isomorphism $(L, \{m_n\}_{n\geq 1}) \xrightarrow{f} (L', \{m'_n\}_{n\geq 1})$ admits a* quasi-inverse *i.e. there exists an L_∞ quasi-isomorphism $(L', \{m'_n\}_{n\geq 1}) \xrightarrow{g} (L, \{m_n\}_{n\geq 1})$ such that $g_{1*} = (f_{1*})^{-1}$.*

A.2.2 Morphisms to a dG Lie algebra

This subsection gives the explicit conditions satisfied by an L_∞ morphism from an L_∞ algebra to a dG Lie algebra. This is the case relevant for section 3 of the paper.

Proposition 4. [26] *Given an L_∞ algebra $(L, \{m_n\}_{n\geq 1})$ and a dG Lie algebra $(L', Q, [.,.])$, an L_∞* morphism *$f = \{f_n\}_{n\geq 1}$ from L to L' amounts to a collection of maps $f_n : \Lambda^n L \to L'$ which are homogeneous of degree $1-n$ and satisfy the conditions:*

$$\begin{aligned}
&Qf_n(u_1,\ldots,u_n) + \sum_{j+k=n+1}\sum_{\sigma\in Sh(k,n)} (-1)^{k(j-1)+1}\chi(\sigma) f_j(m_k(u_{\sigma(1)},\ldots,u_{\sigma(k)}), u_{\sigma(k+1)},\ldots,u_{\sigma(n)}) = \\
&\quad = \sum_{s+t=n}\sum_{\substack{\tau\in Sh(s,n)\\ \tau(1)<\tau(s+1)}} (-1)^{s+(t-1)\sum_{p=1}^{s}|u_{\tau(p)}|}\chi(\tau)[f_s(u_{\tau(1)},\ldots,u_{\tau(s)}), f_t(u_{\tau(s+1)},\ldots,u_{\tau(n)})]\,.
\end{aligned} \tag{A.23}$$

for $n \geq 1$.

A.2.3 The deformation functor of an L_∞ algebra

Definition 10. [9, 12] *Given an L_∞ algebra $(L, \{m_n\})$, its (unextended)* deformation functor *Def^0_L is the functor from the category of local Artin algebras to the category of sets which associates to the algebra B the moduli space:*

$$Def_L(B) = \left\{\phi \in (L\otimes m_B)^1 \,|\, \sum_{n\geq 1} \frac{(-1)^{n(n+1)/2}}{n!} m_n(\phi^{\otimes n}) = 0\right\}/\mathcal{G}_L(B) \tag{A.24}$$

of solutions to the 'homotopy Maurer-Cartan equation' taken modulo the action $\mathcal{G}_L(B)$ generated by infinitesimal transformations of the form:

$$\phi \to \phi' = \phi - \sum_{n\geq 1} \frac{(-1)^{n(n-1)/2}}{(n-1)!} m_n(\alpha \otimes \phi^{\otimes n-1}) \,, \tag{A.25}$$

where $\alpha \in (L \otimes m_B)^0$. Here m_B is the maximal ideal of B.

Observations. 1. The technical device of tensoring with an Artin algebra leads to *formal deformations*, thereby eliminating problems of convergence. The associated formal moduli space is obtained by representing the deformation functor.

2. The fact that infinitesimal gauge transformations of the form (A.25) preserve the homotopy Maurer-Cartan equation is checked by direct computation in appendix B. The explicit form of these transformations differs from that given in [12] through the sign factors.

Theorem 2. [9, 41] *If two L_∞ algebras L and L' are quasi-isomorphic, then their deformation functors are equivalent, $Def^0_L \approx Def^0_{L'}$.*

A.3 Commutator algebra of an A_∞ algebra

Definition 11. [26]

Given an A_∞ algebra $(A, \{r_n\})$, we define new products m_n by graded antisymmetrization of r_n:

$$m_n(u_1 \dots u_n) = \sum_{\sigma\in S_n} \chi(\sigma, u_1 \dots u_n) r_n(u_{\sigma(1)} \dots u_{\sigma(n)}) \,. \tag{A.26}$$

It is then easy to check that $(A, \{m_n\})$ is an L_∞ algebra, called the commutator algebra *of $(A, \{r_n\})$. If the higher products r_n $(n \geq 3)$ vanish, so that A is a differential graded associative algebra, then its commutator algebra is simply the associated differential graded Lie algebra.*

Dual description [26]. Let S be the injective coalgebra morphism from $S(A[1])$ to $T(A[1])$ defined through:

$$S(w_1 \odot \cdots \odot w_n) = \sum_{\sigma\in S_n} \epsilon(\sigma, w_1 \dots w_n) w_\sigma(1) \otimes \ldots \otimes w_{\sigma(n)} \,. \tag{A.27}$$

This map allows us to view $S(A[1])$ as a sub-coalgebra of $T(A[1])$. One then has the following:

Proposition 5. [26] *The A_∞ codifferential ∂ preserves the subspace $\mathrm{Im}\, S$ of $T(A[1])$. In fact, there exists a unique degree one codifferential δ on $S(A[1])$ such that:*

$$\partial S = S\delta \,. \tag{A.28}$$

Moreover, the codifferential δ defines the commutator L_∞ structure (A.26).

Proposition 6. *Given an A_∞ morphism $f = \{f_n\}_{n\geq 1}$ between two A_∞ algebras $(A, \{r_n\})$ and $(A', \{r'_n\})$, we define its (graded) antisymmetrization $g = \{g_n\}_{n\geq 1}$ through:*

$$g_n(u_1 \dots u_n) = \sum_{\sigma\in S_n} \chi(\sigma, u_1 \dots u_n) f_n(u_{\sigma(1)} \dots u_{\sigma(n)}) \,. \tag{A.29}$$

Then g is an L_∞ morphism between the commutator L_∞ algebras $(A, \{m_n\}_{n\geq 1})$ and $(A', \{m'_n\}_{n\geq 1})$. Moreover, if f is a quasi-isomorphism, then so is g.

Proof. Consider the associated map $F : T(A[1]) \to T(A'[1])$, which satisfies $\partial' F = F\partial$. It is easy to see that (A.29) correspond to the unique map $G : S(A[1]) \to S(A'[1])$ with the property:

$$S'G = FS \,. \tag{A.30}$$

We have to show that $\delta' G = G\delta$. In view of injectivity of S', it suffices to show that $S'\delta'G = S'G\delta$. This follows from the chain of equalities:

$$S'\delta'G = \partial' S'G = \partial' FS = F\partial S = FS\delta = S'G\delta \,. \tag{A.31}$$

If f is a quasi-isomorphism, then f_1 induces an isomorphism on cohomology. It is clear from (A.29) that $g_1 = f_1$, hence g is a quasi-isomorphism as well.

B. Infinitesimal gauge transformations in an L_∞ algebra

In this appendix we give a direct proof of the fact that transformations (A.25) preserve the homotopy Maurer-Cartan equation (3.17). The explicit form of these transformations is of independent interest, for example for a better understanding of moduli spaces in closed string field theory [21]. Let us consider a degree one solution ϕ of the homotopy Maurer-Cartan equations:

$$\sum_{l\geq 1} \frac{(-1)^{l(l+1)/2}}{l!} m_l(\phi^{\otimes l}) = 0 \,, \tag{B.1}$$

and an infinitesimal gauge variation of the type (A.25):

$$\delta\phi = -\sum_{k\geq 1} \frac{(-1)^{k(k-1)/2}}{(k-1)!} m_k(\alpha \otimes \phi^{\otimes k-1}) \,, \tag{B.2}$$

where α is a degree zero element of L. Then $\phi + \delta\phi$ satisfies the homotopy Maurer-Cartan equation to first order in α provided that the variation δM of the left hand side of (B.1) vanishes. This first order variation is given by:

$$\delta M = \sum_{l\geq 1} \frac{(-1)^{l(l+1)/2}}{(l-1)!} m_l(\delta\phi, \phi^{\otimes l-1}) \,. \tag{B.3}$$

Upon substituting (B.2), this becomes:

$$\delta M = -\sum_{k,l\geq 1} \frac{(-1)^{k(k-1)/2+l(l+1)/2}}{(k-1)!(l-1)!} m_l(m_k(\alpha\otimes\phi^{\otimes k-1}), \phi^{\otimes l-1})\,. \tag{B.4}$$

We will show that vanishing of this quantity follows from the homotopy Lie identities (3.10). For this, we start by re-writing the latter in the form:

$$\sum_{k+l=n+1}\sum_{\sigma\in Sh(k,n)} (-1)^{l(k-1)}\chi(\sigma;u_1\dots u_n) m_l(m_k(u_{\sigma(1)}\dots u_{\sigma(k)}), u_{\sigma(k+1)}\dots u_{\sigma(n)}) = 0\,, \tag{B.5}$$

which is obtained upon multiplying (3.10) with $(-1)^{n+1} = (-1)^{k+l}$. This allows us to trade the sign factor $(-1)^{k(l-1)}$ in (3.10) for $(-1)^{l(k-1)} = (-1)^{k(l-1)}(-1)^{n+1}$, a trick which will be useful later.

We now apply (B.5) to the elements $u_1 = \alpha$ and $u_2 = \dots = u_n = \phi$. We want to extract the explicit form of the unsigned and signed summands $s(k,n) = m_l(m_k(u_{\sigma(1)}\dots u_{\sigma(k)}), u_{\sigma(k+1)}\dots u_{\sigma(n)})$ and $S(k,n) = \chi(\sigma;u_1\dots u_n)s(k,n)$ for these values of u_j. For this, note that each (k,n)-shuffle σ determines a partition $\{1\dots n\} = I\sqcup J$ of the set $\{1\dots n\}$ through:

$$I = \{\sigma(1)\dots\sigma(k)\}\,, \qquad J = \{\sigma(k+1)\dots\sigma(n)\}\,. \tag{B.6}$$

Since 1 must belong to precisely one of these two sets, and since $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(n)$, we can divide the set of (k,n)-shuffles into two disjoint subsets:

1. The set $Sh_+(k,n)$, of shuffles for which 1 belongs to I, i.e. $\sigma(1) = 1$
2. The set $Sh_-(k,n)$, of shuffles for which 1 belongs to J, i.e. $\sigma(k+1) = 1$.

It is clear that $Sh_+(k,n)$ contains C_{n-1}^{k-1} elements, while $Sh_-(k,n)$ has cardinality C_{n-1}^{k} (C_a^b denotes the number of ways of choosing b out of a elements). Moreover, if the shuffle σ belongs to $Sh_+(k,n)$, then it easy to check that:

$$s(k,n) = m_l(m_k(\alpha,\phi^{\otimes k-1}),\ \phi^{\otimes l-1}) \qquad \text{and} \qquad S(k,n) = s(k,n)\,, \tag{B.7}$$

while if σ belongs to $Sh_-(k,n)$ then one has:

$$s(k,n) = (-1)^l\, m_l(m_k(\phi^{\otimes k}),\phi^{\otimes l-2},\alpha) \qquad \text{and} \qquad S(k,n) = (-1)^k\, s(k,n)\,. \tag{B.8}$$

Upon substituting this in (B.5) and multiplying everything by $\frac{(-1)^{n(n+1)/2}}{(n-1)!}$, we obtain:

$$\Sigma_+(n) + \Sigma_-(n) = 0\,, \tag{B.9}$$

where:

$$\Sigma_+(n) = \sum_{k+l=n+1} \frac{(-1)^{k(k-1)/2+l(l+1)/2}}{(k-1)!(l-1)!} m_l(m_k(\alpha,\phi^{\otimes k-1}),\phi^{\otimes l-1})\,. \tag{B.10}$$

and

$$\Sigma_-(n) = \sum_{k+l=n+1} \frac{(-1)^{k(k+1)/2+l(l-1)/2}}{k!(l-2)!} m_l(m_k(\phi^{\otimes k}), \phi^{\otimes l-2}, \alpha)\,. \tag{B.11}$$

To arrive at these expressions, we used the identities:

$$\frac{k(k-1)}{2} + \frac{l(l+1)}{2} + l(k-1) = \frac{n(n+1)}{2} \tag{B.12}$$

and

$$\frac{k(k+1)}{2} + \frac{l(l-1)}{2} + k(l-1) = \frac{n(n+1)}{2} \tag{B.13}$$

in order to re-write the various sign factors. Comparing eqs. (B.10) and (B.4) shows that:

$$\sum_{n\geq 1} \Sigma_+(n) = -\delta M\,. \tag{B.14}$$

On the other hand, the homotopy Maurer-Cartan equation (B.1) implies:

$$\sum_{n\geq 1} \Sigma_-(n) = 0\,. \tag{B.15}$$

Hence summing the identities (B.9) over n gives the desired conclusion:

$$\delta M = 0\,. \tag{B.16}$$

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